

Self-consistent theory of rupture by progressive diffuse damage

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We analyze a self-consistent theory of crack growth controlled by a cumulative damage variable $d(t)$ dependent on stress history, in the quasistatic regime where the sound wave velocity is taken as infinite. Depending upon the damage exponent m , which controls the rate of damage $dd/dt \propto \sigma^m$ as a function of local stress σ , we find two regimes. For $0 < m < 2$, the model predicts a finite-time singularity. This retrieves previous results by Zobnin for $m = 1$ and by Bradley and Wu for $0 < m < 2$. To improve on this self-consistent theory which neglects the dependence of stress on damage, we apply the functional renormalization method of Yukalov and Gluzman and find that divergences are replaced by singularities with exponents in agreement with those found in acoustic emission experiments. For $m \geq 2$, the rupture dynamics is not defined without the introduction of a regularizing scheme. We investigate three regularization schemes involving, respectively, a saturation of damage, a minimum distance of approach to the crack tip, and a fixed stress maximum. In the first and third schemes, the finite-time singularity is replaced by a crack dynamics defined for all times but which is controlled by either the existence of a microscopic scale at which the stress is regularized or by the maximum sustainable stress. In the second scheme, a finite-time singularity is again found. In the first two schemes within this regime $m \geq 2$, the theory has no continuous limit.

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I. INTRODUCTION

The fracture of materials is a catastrophic phenomenon of considerable technological and scientific importance. Despite the large amount of experimental data and the considerable effort that has been undertaken by material scientists [1], many questions about fracture remain standing. There is no comprehensive understanding of rupture phenomena but only a partial classification in restricted and relatively simple situations. This lack of fundamental understanding is reflected in the absence of reliable prediction methods for rupture based on a suitable monitoring of the stressed system.

Some progress has been made in recent years in the physics community. Based on analogies with phase transitions, several groups [2–13] have proposed that, in heterogeneous materials with disorder such as fiber composites, rocks, concrete under compression, and materials with large distributed residual stresses, rupture is a genuine critical point, i.e., the culmination of a self-organization of diffuse damage and microcracking characterized by power-law signatures. Experiments [3,8,10,12,13], numerical simulations [6,7,9,11], and theory [9] confirm this concept.

As a signature of criticality, acoustic emissions radiated during loading exhibit an acceleration of their rate close to rupture [3,10,13]. Specifically, under a constant stress rate, the cumulative acoustic energy $E(t)$ released up to time t can be expressed as

$$E(t) = E_0 - B(t_c - t)^\alpha, \quad (1)$$

with $B > 0$ and $0 < \alpha < 1$. Expression (1) corresponds to a rate dE/dt of acoustic energy release diverging at the critical rupture time t_c with a negative exponent $\alpha - 1$. This behavior (1) has been at the basis of previous claims that rupture is a critical phenomenon. In addition, this power law (1) as well as extensions with log-periodic corrections have been suggested to be useful for prediction [14,3,4,12,13].

Our purpose here is to present, extend, and analyze a simple self-consistent model of damage that predicts a behavior similar to Eq. (1). We explore its different regimes and then improve on its “mean-field” version which predicts an unrealistic finite-time singularity. In this goal, we propose to use the general functional renormalization approach developed by Yukalov and Gluzman to cure this anomaly. We show how this technique allows us to change an unrealistic singularity into the observed behavior (1) with a reasonable exponent $\alpha = 1/2$, without introduction of any extra parameters in the theory.

II. CUMULATIVE DAMAGE MODEL

Initially introduced as a global “mean-field” (uniform) description of the global deterioration of the system at the macroscopic scale [15], the concept of “damage” has been extended at the mesoscopic scale to describe the heterogeneity and spatial variability of damage in different locations within the material [6,16–18]. We use the formulation of Zobnin [19] and Rabotnov [20] to show how it leads naturally to a finite-time singularity. We first recall briefly the integral formulation of Rabotnov [20] (pps. 166–170) and then transform it in differential form to exhibit the fundamentally nonlinear geometrical origin of the singularity.

A material is subjected to a stress σ_0 at large scale and

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each point r within it carries a damage variable $d(r,t)$. When d reaches the threshold d^* at some location, this local domain is no longer able to sustain stress and a microcrack appears, leading to a redistribution of the stress field around it according to the laws of elasticity. From a physical point of view, a solid is considered to be damaged if some of the bonds connecting parts of its microstructure are missing. The introduction of the damage variable involves a local averaging of the effect produced mostly by microcracks. The damage variable is a continuum dimensionless quantity which measures the cumulative effect which the microcracks, as well as other microdefects, have on the macroscopic response of the system. The effect of many microdefects can be described analytically as we do here by a damage variable only when the material is statistically homogeneous [21].

Within a continuous description, the damage variable is taken to vary smoothly and continuously in space and time, so that we can define the local damage $d(r,t)$ at point r at time t . Following previous works [15,17,6,16,18–20], the local damage variable $d(r,t)$ is taken to evolve in time according to

$$\frac{d(d)}{dt} = \frac{1}{t_{\text{charac}}} \left[\frac{\sigma(r,t)}{\sigma_0} \right]^m. \quad (2)$$

$\sigma(r,t)$ is the local stress field at point r at time t . Here σ_0 is a characteristic stress beyond which the rate of damage becomes large. t_{charac} is a characteristic time scale controlling the rate of damage for a given stress condition. m is the so-called damage exponent which can span values from 0 to close to ∞ depending upon the material. A small value of m represents a more “ductile”-like material undergoing broadly distributed damage before the incipient rupture. A large value of m describes a brittle material with a localized damage mostly developing in the most-stressed regions within the material. In the discrete two-dimensional (2D) models of Refs. [6,16], it was indeed shown that rupture reduces to the percolation model in the limit $m \rightarrow 0$. In the other limit $m \rightarrow \infty$, rupture occurs through a one-crack mechanism.

It is convenient to work with dimensionless variables. We thus make the transformation $\sigma(r,t)/\sigma_0 \rightarrow \sigma(r,t)$ and $t/t_{\text{charac}} \rightarrow t$ such that Eq. (2) reads

$$\frac{d(d)}{dt} = [\sigma(r,t)]^m. \quad (3)$$

Following Rabotnov [20], we assume that a major crack dominates the rupture process. If only one crack is present within the system, the stress $\sigma(r,t)$ is easily calculated. Considering only the possibility of a linear straight crack of half-length $a(t)$ advancing within the material at velocity da/dt (see [22] for generalizations to self-affine crack geometries), it is enough to calculate the stress field on the points ahead of the crack to fully characterize the rupture dynamics. For a planar elastic material subjected to a uniformly distributed antiplane stress at infinity with a crack lying on the y axis between $-a(t)$ and $a(t)$, the stress field at point y on the y axis beyond the crack tip is [23]

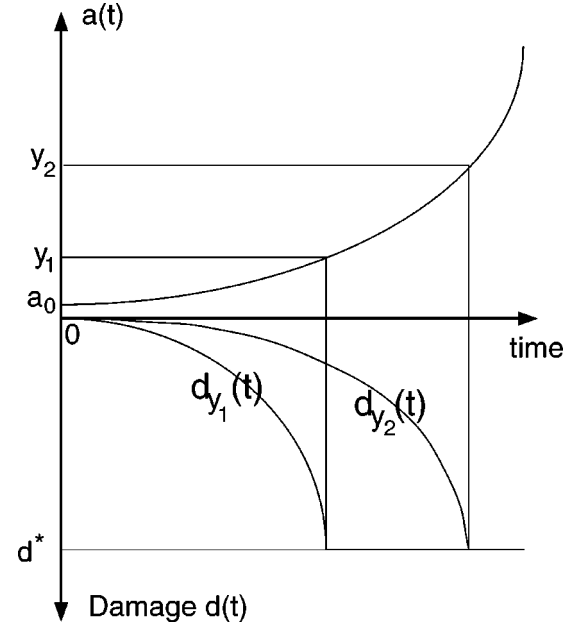


FIG. 1. Illustration of the law governing the growth of the crack: the dynamics of its length $a(t)$ is obtained from the self-consistent condition that the time it takes from a point at y , at the distance $y - a(\tau)$ from the crack tip at time τ , for its damage to reach the rupture threshold d^* be exactly equal to the time taken for the crack to grow from size $a(\tau)$ to the size $a(t) = y$ so that its tip reaches the point y exactly when it ruptures.

$$\sigma(y,t) = \frac{2\sigma_0}{3} \frac{y}{\sqrt{y^2 - [a(t)]^2}}. \quad (4)$$

The mean-field approximation made in this first version of the model consists in assuming that the stress field is not modified by the nonvanishing and nonuniform damage field. This means that the elastic coefficients are taken constant and independent of the progressive damage, except of course when the damage reaches its rupture threshold d^* .

The law describing the growth of the crack, i.e., the dynamics $a(t)$, is obtained from the self-consistent condition that the time it takes from a point at y , at the distance $y - a(\tau)$ from the crack tip at time τ , for its damage to reach the rupture threshold d^* is exactly equal to the time taken for the crack to grow from size $a(\tau)$ to the size $a(t) = y$ so that its tip reaches the point y exactly when it ruptures. This is illustrated in Fig. 1. This self-consistent condition is the only one that is fully consistent with the continuous description, without adding any additional ingredient. It embodies the time delay necessary for the damage at one point ahead of the crack tip to grow up to the rupture point which coincides with the arrival of the crack tip. Other conditions are considered below which introduce characteristic length or time scales.

Mathematically, this self-consistent condition is that the integral of Eq. (3) from time 0 at which the preexisting damage was 0 until time t at which the crack tip passes through y is such that d reaches exactly the threshold d^* at the time t . Two conditions must thus be verified simultaneously: (i)

$a(t)=y$ (the crack tip reaches point y) and (ii) $d(y,t)=d^*$ (the damage at y reaches the rupture threshold).

III. LINEAR DAMAGE LAW: $m=1$

We first consider the linear damage law $m=1$ corresponding to the initial formulation of Zobnin [19]. This case has also been investigated and solved in [24] in the context of crack growth due to electromigration rather than mechanical stress (current plays the role of the stress and, in the antiplane case studied here, the two problems are formally identical). This model is particularly interesting since it allows for both an exact solution and an exact renormalization in the functional renormalization scheme [34]. It also provides a benchmark for approximate solutions in the general case $0 < m < 2$ as we discuss below.

We now proceed to give the equation for the crack dynamics and its solution. By integration of Eq. (3), the two self-consistent conditions expressed for the case $m=1$ lead to

$$\int_0^t d\tau \frac{2\sigma_0}{3} \frac{a(t)}{\sqrt{[a(t)]^2 - [a(\tau)]^2}} = d^*, \quad (5)$$

where the loading stress σ_0 can depend on time. The solution of this integral equation provides the time evolution $a(t)$ of the macrocrack. To get it explicitly, we set

$$z = [a(t)]^2 \quad \text{and} \quad \zeta = [a(\tau)]^2. \quad (6)$$

Changing the variable of integration from τ to ζ gives

$$\int_{z_0}^z d\zeta \sigma_0 \frac{(d\tau/d\zeta)}{\sqrt{z-\zeta}} = \frac{3}{2} \frac{d^*}{\sqrt{z}}. \quad (7)$$

This equation (7) is an Abel equation with index $-1/2$, involving a fractional integral operator [25]. Defining the Abel operator I_α^* acting on the function $f(t)$ as

$$I_\alpha^* \{f\} = \int_0^t \frac{(t-s)^\alpha}{\Gamma(1+\alpha)} f(s) ds, \quad (8)$$

the product of two such Abel operators is

$$\begin{aligned} I_\alpha^* I_\beta^* \{f\} &= I_\alpha^* \{I_\beta^* \{f\}\} = \int_0^t d\tau \frac{(t-\tau)^\alpha}{\Gamma(1+\alpha)} \int_0^\tau \frac{(\tau-s)^\beta}{\Gamma(1+\beta)} f(s) ds \\ &= \int_0^t ds f(s) \frac{1}{\Gamma(1+\alpha)\Gamma(1+\beta)} \int_s^t d\tau (t-\tau)^\alpha (\tau-s)^\beta \\ &= \int_0^t ds f(s) \frac{(t-s)^{\alpha+\beta+1}}{\Gamma(2+\alpha+\beta)}. \end{aligned} \quad (9)$$

This shows that

$$I_\alpha^* I_\beta^* \{f\} = I_{\alpha+\beta+1}^* \{f\}. \quad (10)$$

We thus see that $I_\alpha^* I_{-1-\alpha}^* \{f\} = I_0^*$, which is nothing but the integral operator. The inverse of the Abel operator I_α^* is thus $(d/dt)(I_{-1-\alpha}^*)$. Applying this result to Eq. (7), we find

$$\frac{d\tau}{dz} = \frac{3}{2} \frac{d^*}{\pi\sigma_0} \frac{d}{dz} \int_{z_0}^z \frac{d\zeta}{\sqrt{\zeta(z-\zeta)}}. \quad (11)$$

Calculating the integral on the right-hand side (RHS) of Eq. (11), performing the derivative, and inverting to get $dz/d\tau$, we get

$$\frac{dz}{d\tau} = \frac{2\pi\sigma_0}{3d^*} \frac{z\sqrt{z-z_0}}{\sqrt{z_0}}. \quad (12)$$

Replacing z by $[a(t)]^2$ leads to the differential equation for the crack half-length $a(t)$:

$$\frac{da}{dt} = \frac{\pi\sigma_0}{3d^*} a \sqrt{\left(\frac{a}{a_0}\right)^2 - 1}, \quad (13)$$

which is exactly equivalent to the self-consistent integral equation (5). It is remarkable that the local growth equation (13) embodies exactly the same physics as the long-term memory integral (5).

For simplicity, let us take the loading stress σ_0 constant. This situation is generic of experiments measuring the lifetime of structures under a constant load. At sufficiently long times for which $a(t) \gg a_0$, expression (13) reduces to

$$\frac{da}{dt} \approx \frac{\pi\sigma_0}{3a_0 d^*} a^2. \quad (14)$$

Equation (14) is characteristic of a solution going to infinity in finite time. Indeed, we can write (14) as $da/dt \propto r a$, with a growth rate $r \propto a$. The generic consequence of a power-law acceleration in the growth rate $r \propto a^\delta$ with $\delta > 0$ is the appearance of a singularity in finite time:

$$a(t) \propto (t_c - t)^{-\beta}, \quad \text{with} \quad \beta = \frac{1}{\delta} \quad \text{and} \quad t \text{ close to } t_c. \quad (15)$$

Equation (14) is said to have a ‘‘spontaneous’’ or ‘‘movable’’ singularity at the critical time t_c [26], the critical time t_c being determined by the constant of integration, i.e., the initial condition $a(t=0) = a_0$. Note the intriguing fact that the $(t_c - t)^{-1}$ singularity appears as the solution of a linear mechanical problem. The source of the quadratic nonlinearity is the nonlocal geometrical condition that the delayed action of the stress field on the cumulative damage should coincide exactly with the passage of the crack tip. The nonlinear finite-time singularity has thus fundamentally a nonlocal geometrical origin or, alternatively, can be seen to result from a long-term memory effect.

The exact solution of Eq. (13) is easily obtained by integration:

$$a(t) = \frac{a_0}{\cos[(\pi\sigma_0/3d^*)t]}. \quad (16)$$

This retrieves the solution obtained by Zobnin [19] and Rabotnov [20]. We verify directly that the singularity occurs when the cosine goes to zero, i.e., when the argument reaches $\pi/2$, i.e., for $t_c = 3d^*/2\sigma_0$. Since the cosine vanishes linearly with time, this recovers the asymptotics (15) with the exponent $\beta = -1$, as predicted by the asymptotic equation (14).

Of course, the singularities will not occur in a physical system as the crack tip cannot go to an infinite velocity. As the crack tip accelerates to a velocity which is no longer small compared to the sound wave velocity, the present theory has to be modified to include elastodynamic effects. We stress, however, that this does not invalidate the present treatment, which should be understood as applying to the early stage of the quasistatic crack growth. Here, the term ‘quasistatic’ refers to the approximation in which the finiteness of the sound wave velocity is neglected. The growth of a crack can thus be classified into two regimes: the first early one described here is controlled by the physics of quasistatic damage; the second regime is fully dynamical and the crack tip velocity saturates at a velocity of the order of the sound wave velocity. This classification is well known and has for instance been used successfully in the Ruina-Dieterich theory of frictional rupture [27].

IV. NONLINEAR DAMAGE LAW WITH FINITE-TIME SINGULARITY: $0 < m < 2$

A. Derivation of the differential equation for the crack dynamics

The case where $0 < m < 2$ can be similarly treated and our results here extend those of Zobnin [19] and Rabotnov [20]. Our results retrieve those found in [28], obtained in the context of crack growth due to electromigration. For completeness and coherence in notation, we briefly present the method and the results which are more focused on the finite-time singularity.

For simplicity, we impose σ_0 constant. Integrating Eq. (3) and applying the self-consistent conditions (i) and (ii) given at the end of Sec. II leads to

$$\int_0^t d\tau \left(\frac{2\sigma_0}{3} \right)^m \frac{[a(t)]^m}{\{[a(t)]^2 - [a(\tau)]^2\}^{m/2}} = d^*. \quad (17)$$

We set again the change of variables (6) and changing the variable of integration from τ to ζ gives

$$\int_{z_0}^z d\zeta \frac{(d\tau/d\zeta)}{[z-\zeta]^{m/2}} = \left(\frac{3}{2\sigma_0} \right)^m \frac{d^*}{z^{m/2}}. \quad (18)$$

This equation (7) is again an Abel equation with index $-m/2$ if $0 < m < 2$.

In order to transform it into differential form, we could use the formalism of Abel operators. We choose a more transparent and direct approach which is closely related. First, we multiply both sides of Eq. (18) by $1/(y-z)^{1-m/2}$ and integrate over z from z_0 to y :

$$\begin{aligned} \int_{z_0}^y dz \left(\int_{z_0}^z d\zeta \frac{(d\tau/d\zeta)}{(z-\zeta)^{m/2}(y-z)^{1-m/2}} \right) \\ = \left(\frac{3}{2\sigma_0} \right)^m d^* \int_{z_0}^y \frac{dz}{z^{m/2}(y-z)^{1-m/2}}. \end{aligned} \quad (19)$$

Changing the order of integration on the LHS leads to

$$\begin{aligned} \int_{z_0}^z d\zeta \frac{d\tau}{d\zeta} \left[\int_{\zeta}^y \frac{dz}{(z-\zeta)^{m/2}(y-z)^{1-m/2}} \right] \\ = \left(\frac{3}{2\sigma_0} \right)^m d^* \int_{z_0}^y \frac{dz}{z^{m/2}(y-z)^{1-m/2}}, \end{aligned} \quad (20)$$

where we have used the equality of the triangle $\int_{z_0}^y dz \int_{z_0}^z d\zeta = \int_{z_0}^z d\zeta \int_{\zeta}^y dz$.

The integral in the square brackets on the LHS of Eq. (20) can be expressed through the Euler beta function $B(a, b)$:

$$\begin{aligned} \int_{\zeta}^y \frac{dz}{(z-\zeta)^{m/2}(y-z)^{1-m/2}} \\ = B\left(1 - \frac{m}{2}, \frac{m}{2}\right) = \frac{\Gamma\left(1 - \frac{m}{2}\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(1 - \frac{m}{2} + \frac{m}{2}\right)} = \frac{\pi}{\sin\left(m\frac{\pi}{2}\right)}. \end{aligned} \quad (21)$$

We thus obtain

$$\frac{\pi}{\sin\left(m\frac{\pi}{2}\right)} \int_{z_0}^z d\zeta \frac{d\tau}{d\zeta} = \left(\frac{3}{2\sigma_0} \right)^m d^* \int_{z_0}^y \frac{dz}{z^{m/2}(y-z)^{1-m/2}}. \quad (22)$$

After differentiation with respect to z , we get

$$\frac{d\tau}{dz} = \left(\frac{3}{2\sigma_0} \right)^m d^* \frac{\sin\left(m\frac{\pi}{2}\right)}{\pi} \frac{d}{dz} \int_{z_0}^z \frac{d\zeta}{\zeta^{m/2}(z-\zeta)^{1-m/2}}. \quad (23)$$

B. Asymptotic solution close to the finite-time singularity

The solution of Eq. (23) can be obtained for large crack sizes $a(t)$, i.e., large z . In this goal, we replace the term $(z-\zeta)^{1-m/2}$ in the integral on the RHS of Eq. (23) by $z^{1-m/2}$, neglecting ζ compared to z . Intuitively, this is justified over the whole domain of integration because the contribution from the domain where ζ is not negligible compared to z is finite, since the power $1-m/2$ is less than 1, corresponding to an integrable singularity.

With this approximation, the integral can be performed, the derivative taken, and after inverting, we get

$$\left(\frac{3}{2\sigma_0}\right)^m d^* \frac{\sin\left(m\frac{\pi}{2}\right)}{\pi} \frac{dz}{d\tau} = z_0^{-1+m/2} z^{2-m/2}. \quad (24)$$

Using $a = \sqrt{z}$ as defined in Eq. (6), we obtain

$$a(t) = \frac{a_0}{\left(1 - \frac{t}{t_c}\right)^\beta}, \quad (25)$$

where

$$t_c = \left(\frac{3}{2\sigma_0}\right)^m d^* \frac{2 \sin\left(m\frac{\pi}{2}\right)}{\pi(2-m)} \quad (26)$$

and

$$\beta = \frac{1}{2-m}. \quad (27)$$

Note that the exact asymptotics (15) of the case $m=1$ previously solved exactly is recovered, with the correct exponent $\beta(m=1)=1$ and a rather good approximation of the critical t_c : while the exact value is $t_c = 3d^*/3\sigma_0$, expression (26) predicts $(2/\pi)t_c$, i.e., 36% lower. The critical time t_c as a function of m is smooth with no accident or divergence over the whole interval. In particular, the estimated critical time for the limit $m \rightarrow 2^-$ is equal to $9d^*/4\sigma_0^2$.

In contrast, the exponent z increases from $\beta(m \rightarrow 0^+) = 1/2$ to $+\infty$ as the damage exponent m varies from 0 to 2. The limit $\beta(m \rightarrow 0^+) = 1/2$ can be rationalized as follows. This limit $m \rightarrow 0^+$ corresponds to the situation where damage becomes independent of stress. As a consequence, reintroducing some heterogeneity for instance on the preexisting damage, rupture is then equivalent to percolation, as the parts of the system that break as a function of time are determined by the damage accumulating at the same rate for all point but with different random initial values. In mean-field percolation [29] obtained through the consideration of one-dimensional percolating paths consistent with the present one-crack geometry, the elastic energy under constant load diverges as $(t_c - t)^{-1}$ where 1 is the mean-field value of the exponent for conductivity (which is the same as elasticity in the scalar mode III version of mechanical deformations used here). Since the elastic energy is proportional to the square of the crack length, we get the prediction $a(t) \sim (t_c - t)^{-1/2}$. This reasoning holds if the exponent is a smooth function of disorder and geometry (the present case studied here is a the zero-disorder limit).

The divergence of β at $m=2$ signals a change of regime that we study in the next section.

V. NONLINEAR DAMAGE LAW WITH $m \geq 2$

For $m \geq 2$, the integrals in Eqs. (17) and (18) diverge at $a(\tau) = a(t)$, since the negative power with exponent $m/2$ is no longer integrable. Technically, the main difference be-

tween the cases $m < 2$ and $m \geq 2$ is that we need to regularize the infinity in the expression (21) by introducing some sort of dimensionless cutoff. The important physical message is that the regime where $m \geq 2$ is controlled by a novel physical parameter, which we identify as a length scale associated with the damage law. In other words, the physics of the rupture is inherently controlled by the choice of the cutoff, i.e., by the existence of a microscopic length scale. We could summarize the situation by saying that there is no continuous limit to the theory for $m \geq 2$. This is similar to previous observations obtained in a dynamical theory of rupture front propagation [30]. We now present two ways for regularizing the divergence and thus for obtaining a meaningful theory of rupture.

A. Regularization by damage saturation at a microscopic scale

Before describing the physical content of the regularization we propose, we need to express the problem in a more manageable mathematical form. Since the culprit for the divergence is the integral (21) and the divergence occurs for $\zeta \rightarrow z^-$, we introduce the variable

$$Z = \frac{z - \zeta}{y - \zeta} \quad (28)$$

and rewrite Eq. (21) as

$$\int_{\zeta}^y \frac{dz}{(z - \zeta)^{m/2} (y - z)^{1-m/2}} = \int_0^1 dZZ^{-m/2} (1-Z)^{m/2-1}, \quad (29)$$

which makes it apparent that the divergence is due to $Z^{-m/2}$ at the lower bound 0. It is thus natural to regularize by introducing a dimensionless cutoff $\epsilon > 0$ and replace Eq. (29) by

$$\int_{\epsilon}^1 dZZ^{-m/2} (1-Z)^{m/2-1} \equiv b(m, \epsilon). \quad (30)$$

The function $b(m, \epsilon)$ is such that

$$\lim_{\epsilon \rightarrow 0^+} b(m, \epsilon) = B\left(1 - \frac{m}{2}, \frac{m}{2}\right), \quad \text{for } 0 < m < 2, \quad (31)$$

where the beta function $B(1 - m/2, m/2)$ has been defined in Eq. (21).

In contrast, we have

$$b(m, \epsilon) \sim \frac{1}{\epsilon^{(m-2)/2}}, \quad \text{for } m > 2, \quad (32)$$

and

$$b(m, \epsilon) \sim \frac{1}{\epsilon}, \quad \text{for } m = 2, \quad (33)$$

showing that the divergence of the integral (29) is now encapsulated in the dependence of the factor $b(m, \epsilon)$ on ϵ . This

regularization scheme thus relies on the existence of the definite integral (30), by analogy with the case $m < 2$.

Using the regularization (30), we obtain

$$\frac{d\tau}{dz} = \left(\frac{3}{2\sigma_0}\right)^m \frac{d^*}{b(m, \epsilon)} \frac{d}{dz} \int_{z_0}^z \frac{d\zeta}{\zeta^{m/2}(y-\zeta)^{1-m/2}}, \quad (34)$$

which extends Eq. (11) to the regime $m \geq 2$. Its formal solution obtained in implicit form is

$$t = \left(\frac{3}{2\sigma_0}\right)^m \frac{d^*}{b(m, \epsilon)} \int_{z_0}^z dy \frac{d}{dy} \int_{z_0}^y \frac{d\zeta}{\zeta^{m/2}(y-\zeta)^{1-m/2}}. \quad (35)$$

This regularization scheme allows us to obtain exact solutions for integer m 's. We examine the solutions for $m=2, 3, 4$, and 5 and then the general case. For $m=2$, we have the expression for all times given by

$$z_2(t) = z_0 e^{t/t_{\epsilon,2}}, \quad (36)$$

where

$$t_{\epsilon,2} = \frac{9d^*}{4\sigma_0^2} \frac{1}{\ln \frac{1}{\epsilon}}. \quad (37)$$

For $m=3$, $z_3(t)$ is the solution to

$$\sqrt{(z/z_0)-1} - \tan^{-1}[\sqrt{(z/z_0)-1}] = \frac{t}{t_{\epsilon,3}}, \quad (38)$$

where

$$t_{\epsilon,3} = \frac{27d^*}{8\sigma_0^3 \sqrt{z_0}} \epsilon^{1/2}. \quad (39)$$

For large times, we get

$$z_3(t) \approx z_0 \left(\frac{t}{t_{\epsilon,3}}\right)^2. \quad (40)$$

For $m=4$, $z_4(t)$ is the solution to

$$-\ln(z/z_0) + (z/z_0) - 1 = \frac{t}{t_{\epsilon,4}}, \quad (41)$$

where

$$t_{\epsilon,4} = \frac{81d^*}{16\sigma_0^4} \epsilon. \quad (42)$$

For large times, we get

$$z_4(t) \approx z_0 \frac{t}{t_{\epsilon,4}}. \quad (43)$$

For $m=5$, $z_5(t)$ is the solution to

$$\begin{aligned} & [(z/z_0)-1]^{3/2} - 3[(z/z_0)-1]^{1/2} + 3 \tan^{-1}\{[(z/z_0)-1]^{1/2}\} \\ & = \frac{t}{t_{\epsilon,5}}, \end{aligned} \quad (44)$$

where

$$t_{\epsilon,5} = \frac{243d^*}{32\sigma_0^5} \epsilon^{3/2}. \quad (45)$$

For large times, we get

$$z_5(t) \approx z_0 \left(\frac{t}{t_{\epsilon,5}}\right)^{2/3}. \quad (46)$$

More generally, at large times,

$$z_m(t) \approx z_0 \left(\frac{t}{t_{\epsilon,m}}\right)^{2/(m-2)}, \quad (47)$$

$$a_m(t) \approx a_0 \left(\frac{t}{t_{\epsilon,m}}\right)^{1/(m-2)}, \quad (48)$$

where

$$t_{\epsilon,m} \propto \epsilon^{(m-2)/2} \sim \frac{1}{b(m, \epsilon)}. \quad (49)$$

From these solutions, it is apparent that the dynamics $z(t) \equiv [a(t)]^2$ is controlled by the characteristic time $t_{\epsilon,m}$ defined in Eq. (49). Note that the inverse dependence of $t_{\epsilon,m}$ on $b(m, \epsilon)$ is obvious from the expression (35). As the cutoff $\epsilon \rightarrow 0$, $t_{\epsilon,m} \rightarrow 0$ and the global rupture occurs in vanishing time. The physical explanation of this phenomenon is as follows. For $m \geq 2$, the driving force σ^m of the damage law (3) is so strong close to and at the crack tip that it takes effectively zero time for a point to be brought to the damage threshold. To see this, let us truncate the integral in Eq. (17) such that the upper bound is changed from t to $t - \eta$. The divergence of the integral at the crack tip means that the contribution to the cumulative damage occurring in the time interval from $t - \eta$ to t is larger (actually infinitely larger) than the contribution from time 0 to time $t - \eta$. This means that the progressive damage leading to the acceleration of $a(t)$ for $m < 2$ is replaced by an infinite velocity as soon as we start from a finite crack and do not introduce the finite cutoff length.

This clarifies the physical meaning of the cutoff ϵ defined in Eq. (30). A nonzero ϵ means that the integral over ζ in Eq. (29) does not go all the way up to z . Translated in terms of physical distances, it means that the integral in ζ on the LHS of Eq. (18) also does not go all the way up to z . Physically, this means that the damage on a given point ahead of the crack tip reaches the critical value d^* before the crack tip reaches that point. The value d^* is no longer the rupture threshold but a saturation value. The crack tip dynamics is now determined by the condition that the damage at any given point y reaches this saturation value d^* when the crack tip is at a fixed distance $\propto \epsilon$ from y . This condition embodies

the existence of a microscopic length scale $\propto \epsilon$ such that the damage is no longer defined as smaller scales. Note, however, that the regularization performed in Eq. (30) has been chosen for its mathematical simplicity. It is not related in an obvious way to a clearly defined physical mechanism, such as plasticity, creep, or time delays. It only qualitatively embodies the existence of a finite length scale. This is the reason for our consideration of more physically motivated but more complex mathematically regularization schemes studied below.

The main result of our analysis is that the characteristic time scale $t_{\epsilon,m}$ of the crack dynamics is controlled by the microscopic length scale. The theory has thus fundamentally no continuous limit. It is one of several interesting and important examples in physics where the macroscopic physics is completely controlled by the microscopic physics (the ultraviolet cutoff). This situation is found in many physical problems: for instance, in correlation functions in two-dimensional systems [31] and in nonlinear diffusion [33] as well as in quantum electrodynamics [32]. Note, however, the difference between the last two examples and the former ones: in our rupture problem as well as in the case of correlation functions in 2D systems, the ultraviolet cutoff appears naturally, as an atomic distance, while in the last two cases, there is no meaningful natural cutoff, and hence the necessity to “cover up” divergencies by the “renormalization” procedure [32].

B. Regularization by stress saturation at a microscopic scale

The previous regularization scheme invokes a saturation of the damage at a microscopic length $\propto \epsilon$. Alternatively, the saturation can occur on the stress field, whose mathematical divergence is bound to be rounded off at atomic scales. This provides another regularization scheme. To implement it, we use the continuous expression (4) for all distances from the crack tip down to a regularization length ℓ such that, for distances from the crack tip from 0 to ℓ , the stress is constant equal to $\sigma(\ell)$ given by Eq. (4) with $y = a(t) + \ell$. This regularization is standard in the theory of damage and of plasticity. The idea is that a sufficiently large damage exerts a feedback on the stress field which then departs from its damage-free continuous expression (4). This extension to Rabotnov’s treatment provides a natural way for constructing a self-consistent theory of damage: not only does rupture occur by the cumulative effect of damage, damage has also the effect of smoothing out the mathematical singularity at the crack tip. The cutoff ℓ has the physical meaning of a so-called process zone or damage zone and its introduction is fully consistent with the dynamical damage law (3). However, as we have seen above, for a damage law with an exponent $0 < m < 2$, the resulting dynamics becomes insensitive to the existence of a microscopic length scale in the limit where it is small. In this sense, the regime $0 < m < 2$ is more universal and has a continuous limit.

We propose two models that implement these ideas.

1. Saturation of the stress at a fixed distance to the crack tip

The regularization scheme used here is such that the stress is assumed to saturate at a fixed distance ℓ from the crack

tip. Thus, for times τ up to $t_\ell(t)$, the damage at a fixed point that will be reached by the crack tip at time t is growing under the influence of the stress field created by the crack. From time $t_\ell(t)$ up to time t , the damage is increasing linearly with time, since the stress is assumed constant and equal to the value it reaches at time $t_\ell(t)$. The saturation time $t_\ell(t)$ is determined by the equation

$$a(t) - a(t_\ell) = \ell. \quad (50)$$

In this version of the regularized theory, expression (17) is changed into

$$\int_0^{t_\ell} d\tau \left(\frac{2\sigma_0}{3} \right)^m \frac{[a(t)]^m}{\{[a(t)]^2 - [a(\tau)]^2\}^{m/2}} + \left(\frac{2\sigma_0}{3} \right)^m \frac{[a(t)]^m}{[(2a(t) - \ell)\ell]^{m/2}} (t - t_\ell) = d^*, \quad (51)$$

where t_ℓ is the time at which the crack tip is at the distance ℓ from the position it will have at time t [see Eq. (50)].

Note that we now have two equations for two unknown $a(t)$ and $a(t_\ell)$. The second term on the LHS of Eq. (51) expresses the linear increase in damage from t_ℓ to t under the saturated stress. In this model, ℓ is fixed and t_ℓ adjusts itself. The value of the saturated stress is not a constant but increases as the crack gets larger and larger, since it corresponds to the value at a fixed distance ℓ from the tip of a growing crack.

The change of variables (6) and changing the variable of integration from τ to ζ give

$$\int_{z_0}^{(\sqrt{z}-\ell)^2} d\zeta \frac{(d\tau/d\zeta)}{[z-\zeta]^{m/2}} + \frac{t-t_\ell}{[(2\sqrt{z}-\ell)\ell]^{m/2}} = \left(\frac{3}{2\sigma_0} \right)^m \frac{d^*}{z^{m/2}} \quad \text{with} \quad \sqrt{z} - a(t_\ell) = \ell. \quad (52)$$

Note that $t - t_\ell$ can also be written

$$\tau(z) - \tau((\sqrt{z}-\ell)^2) \approx 2\ell\sqrt{z} \frac{d\tau}{dz} \Big|_z + O(\ell^2). \quad (53)$$

The integral on the LHS of Eq. (52) is analyzed similarly to the previous case (18). We multiply the integral by $1/(y-z)^{1-m/2}$ and integrate over z from $(\sqrt{z_0} + \ell)^2$ to y :

$$\int_{(\sqrt{z_0} + \ell)^2}^y dz \left(\int_{z_0}^{(\sqrt{z}-\ell)^2} d\zeta \frac{(d\tau/d\zeta)}{(z-\zeta)^{m/2}(y-z)^{1-m/2}} \right) = \int_{z_0}^{(\sqrt{y}-\ell)^2} d\zeta \frac{d\tau}{d\zeta} \left[\int_{(\sqrt{\zeta} + \ell)^2}^y \frac{dz}{(z-\zeta)^{m/2}(y-z)^{1-m/2}} \right], \quad (54)$$

where we have used the equality of the triangle $\int_{(\sqrt{z_0} + \ell)^2}^y dz \int_{z_0}^{(\sqrt{z}-\ell)^2} d\zeta = \int_{z_0}^{(\sqrt{y}-\ell)^2} d\zeta \int_{(\sqrt{\zeta} + \ell)^2}^y dz$.

The integral in the brackets on the RHS is the same as in Eq. (30), which defines the function $b(m, \epsilon)$ with

$$\epsilon(\zeta) = \frac{(\sqrt{\zeta} + \ell)^2 - \zeta}{y - \zeta} \approx \frac{2\ell\sqrt{\zeta}}{y - \zeta}. \quad (55)$$

Note that $\epsilon(\zeta)$ is now a function of ζ .

Using Eqs. (53) and (54), expression (52) gives

$$\begin{aligned} & \int_{z_0}^{(\sqrt{y} - \ell)^2} d\zeta \frac{d\tau}{d\zeta} b(m, \epsilon(\zeta)) \\ & + \int_{(\sqrt{z_0} + \ell)^2}^y dz \frac{2\ell\sqrt{z}}{(y-z)^{1-m/2} [(2\sqrt{z} - \ell)\ell]^{m/2}} \frac{d\tau}{dz} \\ & = \left(\frac{3\sqrt{d^*}}{2\sigma_0} \right)^m \int_{(\sqrt{z_0} + \ell)^2}^y \frac{d\zeta}{\zeta^{m/2} (y - \zeta)^{1-m/2}}. \end{aligned} \quad (56)$$

Since

$$b(m, \epsilon(\zeta)) \propto 1/\epsilon^{m/2-1} \propto \left(\frac{2\ell\sqrt{\zeta}}{y - \zeta} \right)^{1-m/2}, \quad (57)$$

we see that the first integral of the LHS of Eq. (56) is negligible compared to the second integral of the LHS of Eq. (56) in the limit of large cracks, i.e., large z . Neglecting ℓ compared to \sqrt{z} in the denominator of the integrand of the second integral of Eq. (56) and equating this second integral to the RHS gives the following equation:

$$\frac{d\tau}{dz} = (2\ell)^{m/2-1} \left(\frac{3\sqrt{d^*}}{2\sigma_0} \right)^m \frac{1}{z^{m/4+1/2}}. \quad (58)$$

For $m > 2$, $m/4 + \frac{1}{2} > 1$ and the solution of Eq. (58) is

$$a(t) \propto \frac{\ell}{(t_c - t)^{2/(m-2)}}, \quad (59)$$

where t_c is determined from the initial size of the crack. The finite-time singularity results from the ever-increasing stress field at the fixed distance ℓ from the crack tip. This solution (59) is qualitatively different from the solution (25) found in the regime $m < 2$ as Eq. (59) depends in a fundamental way upon the existence of the regularization scale ℓ .

2. Saturation by fixing an absolute maximum stress

An alternative prescription for the regularization is that the stress saturates at a constant value σ_{\max} . This is in contrast with the previous regularization scheme where the stress saturates at a value reached at a constant distance, this value thus increasing with the crack length. Expression (51) is then changed into

$$\begin{aligned} & \int_0^{t_\ell} d\tau \left(\frac{2\sigma_0}{3} \right)^m \frac{[a(t)]^m}{\{[a(t)]^2 - [a(\tau)]^2\}^{1/2}} + [\sigma_{\max}]^m (t - t_\ell) \\ & = d^*, \end{aligned} \quad (60)$$

with

$$\frac{2\sigma_0}{3} \frac{a(t)}{\{[a(t)]^2 - [a(t_\ell)]^2\}^{1/2}} = \sigma_{\max}, \quad (61)$$

which is the condition that the stress saturates. It gives

$$[a(t_\ell)]^2 = [a(t)]^2 (1 - A^2), \quad (62)$$

where

$$A = \frac{2\sigma_0}{3\sigma_{\max}}. \quad (63)$$

The equation (60) governing the dynamics of the crack tip can thus be written

$$\frac{2\sigma_0}{3} \int_{z_0}^{(1-A^2)z} d\zeta \frac{d\tau/d\zeta}{(z-\zeta)^{m/2}} + [\sigma_{\max}]^m A^2 z \frac{d\tau}{dz} = d^*, \quad (64)$$

where we have used the expansion

$$\tau(z) - \tau((1-A^2)z) \approx A^2 z \frac{d\tau}{dz} \Big|_z + \mathcal{O}(A^4), \quad (65)$$

valid in the interesting regime $\sigma_{\max} \gg \sigma_0$ giving $A \ll 1$.

The second term on the LHS of Eq. (64) dominates the first integral for large z , as can be checked *a posteriori*. For large crack sizes, the expression (65) can thus be simplified into

$$[\sigma_{\max}]^m A^2 z \frac{d\tau}{dz} = d^*, \quad (66)$$

whose solution is

$$a(t) = a_0 e^{t/t_0}, \quad (67)$$

where

$$t_0 = \frac{2d^*}{A^2 [\sigma_{\max}]^m} = \frac{9d^*}{2\sigma_0^2 [\sigma_{\max}]^{m-2}}. \quad (68)$$

VI. BEYOND THE MEAN-FIELD VERSION BY FUNCTIONAL RENORMALIZATION

Let us restrict our discussion to the case $m = 1$ for which we have the complete analytical solution for the crack dynamics. The solution (16) with its asymptotic behavior (15) is not physically reasonable, as the crack reaches an infinite length in a finite time. The $(t_c - t)^{-1}$ singularity has been found to appear as the consequence of a geometric nonlinearity on an otherwise linearized mechanical problem. In reality, nonlinearity, viscosity, feedback, and spatial heterogeneity of material properties and of cracking should modify the singularity. In addition, the main simplification in the previous approach is to neglect the impact of damage on the elastic coefficients of the material, thus leading to a stress field created by the crack which is identical to the field that

the same *static* crack would generate in an undamaged material. Our hypothesis is that such modification can be deduced by a smooth or regular deformation of the solution previously obtained.

In this goal, we propose to apply the Yukalov-Gluzman functional renormalization method [34] to the series expansion of the solution (16) to obtain the renormalized law that accounts for these effects in a generic sense. Let us first consider the asymptotic power law singularity (15)

$$a(t) = \frac{2a_0}{\pi} (1-x)^{-1}, \quad \text{where } x \equiv t/t_c. \quad (69)$$

The powers x^n in the expansion

$$a(t) = \frac{2a_0}{\pi} (1+x+x^2+x^3+\dots) \quad (70)$$

may be considered as hidden free parameters. Indeed, let us multiply the expansion by x^s . We then have a trial expansion for the solution. For $s=0$, we return to the regular expansion. Such multiplication can be applied repeatedly, for instance, using the functional renormalization method [34]. The idea behind the introduction of the multiplicative (control) function such as the power s in x^s is to deform smoothly the initial functional space of the expression $a(t)$ taken as an approximation to be improved. The condition for the improvement is to obtain a faster and better controlled convergence in the space of the modified functions upon addition of successive terms x^n in the expansion. By this procedure, the dominant poles are eliminated or weakened as a result of a sequential reduction of stress level at each step of the resummation procedure. This corresponds to utilizing the information from the initial series pertaining to the times preceding the critical time t_c , where the level of damage is lower. Thus, the renormalization procedure is performing a mapping from the dynamics at early time far from the critical point to later times closer to the critical time. The stabilization stems from the fact that the information contained in the initial series related to times close to t_c is minimized on the basis that it has an overly destabilizing effect in the description and should be weighted less than the information at earlier times.

At each step of the functional renormalization corresponding to the addition of a new term, we select the renormalized function according to the principle of minimum ‘‘local’’ multiplier, i.e., maximum stability on each substep of the renormalization procedure. Since these multipliers are proportional to the derivative da/dt [34], the principle of minimal multiplier implies a selection of the real-time trajectory of the crack with minimal rate of damage (minimal stresses). In other words, this procedure amounts to improving the theory by allowing the crack to organize and develop so as to choose the most favorable path or dynamics. It can be shown [34] that, at each step, the choice of a formally infinite exponent s corresponds to the minimal multiplier at arbitrary time.

The functional form of a superexponential solution is selected by this procedure: starting from an expansion 1

+ $a_1x + a_2x^2 + a_3x^3 + \dots + a_kx^k$, the renormalized expression is as follows. With the use of the notation

$$b_0 = a_0, \quad b_k = \frac{a_k}{a_{k-1}}, \quad k = 1, 2, \dots, \quad (71)$$

we obtain the *bootstrap self-similar approximant* up to order k ,

$$F_k(x) = b_0 \exp(b_1x \exp\{b_2x \exp[\dots b_{k-1}x \exp(b_kx)]\} \dots), \quad (72)$$

introduced by Yukalov and Gluzman [34].

Let us now apply this result to the case (70) where all the coefficients a_n are equal to 1. The corresponding renormalized approximant replacing the initial input $1/(1-x)$ of the expansion reads

$$F(x) = \exp\{x \exp[x \dots \exp(x)] \dots\}. \quad (73)$$

This embedded exponential series converges to a well-defined function. To determine it, we note that $F(x)$ obeys the recursion relation

$$F_{k+1}(x) = \exp[xF_k(x)]. \quad (74)$$

The fixed point to which these series of approximants converge is thus a solution of

$$F = \exp(xF), \quad (75)$$

noting that $F = \pi a / 2a_0$. The limit $F_\infty(x)$ exists for $-e \leq x \leq e$.

The fixed point $F(x)$ can be shown [26] to be the solution of the equation

$$\frac{dF}{dx} = \frac{F^2}{1-xF}. \quad (76)$$

Searching for a solution in the form of a Taylor series

$$F(x) = \sum_{n=0}^{\infty} y_n x^n, \quad y_0 = 1, \quad (77)$$

we get

$$y_n = \frac{(n+1)^{n-1}}{n!}. \quad (78)$$

Since $n! \approx n^n e^{-n}$, $y_n \approx e^n$ for large n and the generic term $y_n x^n$ in the series (77) is proportional to $(ex)^n$. This shows that the radius of convergence of the series (77) is $1/e$.

$F(x)$ has a singularity when x approaches $1/e$ from below, whose shape is obtained by expansions of expression (75):

$$F(x) = x \rightarrow 1/e e (1 - \sqrt{2e^{3/2} \sqrt{1/e - x}}). \quad (79)$$

Thus, the self-similar functional renormalization has transformed a pole (divergence of a) at $t = t_c$ into a square root singularity (finite a) at a smaller $t = t_c/e$. In this renormalized theory, the crack accelerates up to the time t_c/e at which time its velocity diverges, while the crack is still fi-

nite. This announces the global breakdown. It is interesting that the exponent $1/2$ is close to the value found for acoustic emissions in experiments [3,10,12,13].

We can offer the following physical intuition for this transformation from the solution (15) with $\beta=1$ to $\beta=-1/2$. As the material becomes more and more damaged, the ulterior functional dependence of damage as a function of applied stress is modified. Actually, the series of functional renormalization amounts to effectively evolving or renormalizing the damage law (3) into a succession of effective laws captured by the sequence of approximants, each approximant order corresponding to an increase in the overall damage of the material. Here, we have a mapping between a measure of evolution via the cumulative damage, i.e., a measure of passed time, and the order of the approximants, and thus the distance to the fixed point in the functional space.

Consider now the general case $a(t) \sim (1-x)^{-\beta}$. Expanding in power series, we get

$$(1-x)^{-\beta} = \sum_{n=0}^{\infty} a_n x^n, \quad (80)$$

where

$$a_n = \frac{(n+\beta-1)!}{n!(\beta-1)!}. \quad (81)$$

The Yukalov-Gluzman renormalization scheme gives the superexponential (72) with coefficients b_n given by

$$b_n = (n+\beta-1)/n. \quad (82)$$

Since $b_n \rightarrow 1$ for large n for any β , the fixed point of the approximants is controlled by the same finite square-root singularity of the type (79). Thus, the functional renormalization maps all finite-time singularities with different exponent β on the same universal law $a(t) = a(t_c) - C\sqrt{t_c - t}$, where C is a constant depending in particular on β . Tests of extensions of this scheme for predicting rupture, i.e., for predict-

ing the critical time t_c with respect to the order of the polynomials and as a function of noise, are reported in the companion paper [35].

VII. CONCLUDING REMARKS

Two main regimes have been found for the growth of a crack in a medium obeying the damage law $d(d)/dt = \sigma^m$ [Eq. (3)], where σ is the local stress. For $0 < m < 2$, a preexisting crack grows to infinity in finite time and the divergence occurs as a power-law finite-time singularity. For $m \geq 2$, the solution exists for all times but the characteristic time scale of the crack growth is an increasing function of a microscopic length scale, which is essential for regularizing the otherwise ill-defined problem. This microscopic length scale embodies the physical mechanism(s) by which the mathematical stress singularity at the crack tip of a perfectly sharp crack is rounded off. We have examined two main scenarios, a damage-limited rupture and a stress-limited rupture.

The remarkable behavior of this simple model results from the form of the irreversible damage law, in particular from the fact that any nonvanishing stress increases the damage. Damage at any point is thus a kind of ever-increasing counter of the history of the stress on that point. This feature prevents the existence of stationary solutions of cracks propagating at constant velocities. In contrast, we only obtain “runaways.”

Stationary solutions can be obtained in simple generalizations of the damage law (3), for instance with a stress threshold below which no damage occurs or with a healing or work-hardening term, allowing recovery of the material and a decrease of the damage when the stress is low. Such situations have been investigated in discrete two-dimensional models [16].

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